Multidimensional simple waves in fully relativistic fluids

TALIYA SAHIHI †, *HOMAYOON ESHRAGHI* † *and ALI MAHDIPOUR–SHIRAYEH* ‡

† *School of Physics, Iran University of Science and Technology (IUST), Narmak, Tehran, Iran*

‡ *School of Mathematics, Iran University of Science and Technology (IUST), Narmak, Tehran, Iran*

E-mail:† *taliasah@yahoo.com,eshraghi@iust.ac.ir,* ‡*mahdipour@iust.ac.ir*

A special version of multi–dimensional simple waves given in [G. Boillat, *J. Math. Phys.* **11**, 1482-3 (1970)] and [G.M. Webb, R. Ratkiewicz, M. Brio and G.P. Zank, *J. Plasma Phys.* **59**, 417-460 (1998)] is employed for fully relativistic fluid and plasma flows. Three essential modes: vortex, entropy and sound modes are derived where each of them is different from its nonrelativistic analogue. Vortex and entropy modes are formally solved in both the laboratory frame and the wave frame (co-moving with the wave front) while the sound mode is formally solved only in the wave frame at ultra-relativistic temperatures. In addition, the surface which is the boundary between the permitted and forbidden regions of the solution is introduced and determined. Finally a symmetry analysis is performed for the vortex mode equation up to both point and contact transformations. Fundamental invariants and a form of general solutions of point transformations along with some specific examples are also derived.

Key words: Relativistic Fluids, Multidimensional simple waves, Symmetry analysis

1 Introduction

Investigation of nonlinear phenomena appearing in a very wide area of pure and applied sciences has met extremely extensive progresses and developments. These studies which split into numerical and analytical considerations are essentially and in most cases related to some nonlinear differential equations. In spite of such a huge amount of improvements almost all of these equations are still far from being understood well. Among those nonlinear systems, a few interesting open problems concern the hydrodynamic type of equations governing fluid motions. Especially the Euler and Navier-Stokes equations which reveal a mysterious behavior are being intensively studied in two main considerations: The incompressible motion mostly dealing with vortex dynamics and the compressible flow concerning the appearance of discontinuities shocks. Both of these problems are somehow related to the debate of the regularity of solutions. In the former consideration the aim is to understand the mechanism of occurrence of vortex singularities while in the latter the shock convergence is the most important question[1, 2]. The core of these subjects is the following principal open question: Starting from an initially smooth flow how can we predict the appearance of any kind of discontinuity or singularity in later times? In other words, what is "*hidden*" in the smooth initial conditions which causes the non-smoothness in future? Numerical evidences as well as some analytical investigations in special cases highly confirm the existence of such hidden facts. It seems that these facts are partly related to topological properties and partly to the measure theoretical aspects.

The wide variety of the application of fluid motions causes to deal with all kinds of differential equations namely, elliptic, parabolic and hyperbolic equations. While many systematic treatments have been found and developed for parabolic and elliptic equations, hyperbolic equations are still out of the frame of any well-defined method. These equations posses some characteristic curves or surfaces which naturally have the capability of forming discontinuities. This of course makes the nature of the solution to be "local" which means that the continuous solution may exist only in some part of the space and in some intervals of time. Generally there are many topological, geometrical and analytical unknown features determining the validity of any solution which are very difficult to discover under the present human knowledge and thus new tools are needed.

We believe the best way to obtain some result is to study useful special cases guiding us to more general statements. An excellent and rich class of solutions for compressible ideal flows governed by hyperbolic equations lies in the framework of simple (Riemann) waves which have the capability of shock formation[1, 2] and blow up occurrence[3]. Simple waves constructed on the basis of characteristics are clearly local which were discovered first by Riemann in the 1-D form[4]-[10] and are still the best analytical tools to achieve shock waves[1, 2, 5, 7, 11, 12]. By imposing more restrictions and limitations on these solutions it was possible to build multidimensional simple waves[13]-[18]. Even a more generalization yields double waves and multi-waves which give more advanced solutions with more intensive blow up[18]-[25].

It is obvious that taking into account relativistic effects highly increases the coupling and so the nonlinearity of fluid and plasma motions. Relativistic flows have been known for a long time[27]-[29] and especially they are important is astrophysical and cosmological phenomena. In addition, under recent technical progresses in laser-plasma interaction, plasma accelerators and fusion plasmas, the access to relativistic effects in the laboratory is now very easy. Therefore a great attention has been paid to analyze relativistic flows. Again the study of simple waves plays a very fundamental role as almost the only available nonstationary exact solution with the ability of discontinuity formation.

A very excellent and complete mathematical discussion on one dimensional relativistic MHD simple waves has been reviewed by Shikin[30]. Some solutions of these 1-D simple waves are found in many papers. Although a relativistic 2-D double wave solution solution has been given only for ultra-relativistic fluids[25] but still we observe the missing of a multidimensional simple wave for a fully relativistic flow. This task is the aim of the present note in which the approach of Ref. [17] is employed and generalized.

Physically it is a valid question that why we should consider relativistic fluids while usually at so high temperatures matter is found in the plasma form and so one has to take into account electromagnetic fields leading to MHD equations. However this is not always true because sometimes we deal with neutral fluids like neutron stars. Moreover in the absence of any external magnetic field and when the typical length and time for the non-neutrality of the plasma are sufficiently less than the length and time for macroscopic motions, the plasma can be considered as a neutral fluid with at a very high accuracy. Hence it has sense to consider the ideal relativistic flow here.

In the next section after a brief derivation of relativistic ideal fluid equations, a multidimensional simple wave ansatz is substituted into these equations and various modes and phase velocities relative to the laboratory (fixed) frame are found. In Sec. 3 some solutions for the vortex and entropy modes are given only in the laboratory frame. The presented solutions are very general and formal and a detailed solution is very difficult and needs to determine the initial and boundary conditions precisely. Thus, our solutions are very general including many arbitrary functions. In Sec. 4 the equations are rewritten in the wave frame and again some

simple typical solutions are given for the all three modes vortex, entropy and sound. Especially for the sound since its equations are very complicated in the laboratory frame, it is seen in Sec. 4 that these equations in the wave frame at ultra-relativistic temperatures are simplified and it will be possible to obtain some formal solutions for it. In sec. 5 we investigate symmetry properties and their related topics for the vortex mode equation as a sample equation appearing in our problem. Finally a summary and concluding remarks are given in Sec. 6.

2 Multidimensional simple wave formulation

Relativistic effects in continuum matters in two aspects: Large macroscopic (fluid) velocities and relativistic temperatures at which the mean thermal energy of particles are comparable with their rest energy. Both of these aspects are included in the energy-momentum tensor

$$T_i^k = wu^k u_i - P\delta_i^k \ , \qquad (i,k=0,1,2,3) \tag{1}$$

where $u^j = (\gamma, \gamma\mathbf{v}/c)$ is the contra-variant 4-velocity and thus $u_j = (\gamma, -\gamma\mathbf{v}/c)$ is the co-variant 4-velocity and $w = \varepsilon + P$ in which P is the fluid pressure and ε is the internal energy (including the rest energy) per unit proper volume (unit volume in the inertial frame in which the fluid is momentarily at rest). Therefore w is the enthalpy per unit proper volume. Also $\mathbf{v}$ is the fluid velocity and $\gamma = (1 - v^2/c^2)^{-1/2}$ where c is the speed of light.

Basic equations consist of continuity equation

$$\frac{\partial}{\partial x^i}(nu^i) = 0 \ , \qquad \texttt{or} \qquad \frac{1}{c}\frac{\partial}{\partial t}(\gamma n) + \boldsymbol{\nabla}\cdot(n\gamma\mathbf{v}) = 0 \ , \tag{2}$$

and the vanishing 4-divergence of the energy-momentum tensor

$$\frac{\partial}{\partial x^k}T_i^k = 0 \ , \qquad (i=0,1,2,3) \tag{3}$$

where n is the number density of fluid particles in the proper frame. By virtue of thermodynamic identity $TdS = d(w/n) - dP/n$ (T is the fluid temperature and S is the entropy per unit particle) one can combine Eqs. (2) and (3) to obtain[30]

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}S = 0 \ . \tag{4}$$

This equation can be alternatively considered in place of the zeroth component ($i = 0$) of Eq. (3). Thus our set of equations consists of five equations (2), (4) and the spatial components of (3). This system of course needs a thermodynamical state equation $P = P(S, w)$. However it is found that this system of equations takes a more appropriate form by the use of the following useful transformation[30]

$$\kappa^i = \frac{w}{mnc}u^i \qquad , \qquad \tilde{\rho} = \frac{(mnc)^2}{w} \ , \tag{5}$$

where m is the "mean" rest mass of all particles in the fluid. This transformation makes our final system of equations to

$$\frac{1}{c}\frac{\partial}{\partial t}(\tilde{\rho}\kappa_0) + \boldsymbol{\nabla}\cdot(\tilde{\rho}\boldsymbol{\kappa}) = 0 \ , \tag{6}$$

$$\frac{\kappa_0}{c}\frac{\partial\boldsymbol{\kappa}}{\partial t} + (\boldsymbol{\kappa}\cdot\boldsymbol{\nabla})\boldsymbol{\kappa} = -\frac{1}{\tilde{\rho}}\boldsymbol{\nabla}P \ , \tag{7}$$

$$\frac{\kappa_0}{c}\frac{\partial S}{\partial t} + \boldsymbol{\kappa}\cdot\boldsymbol{\nabla} S = 0 \ , \tag{8}$$

$$P = P(S,\tilde{\rho}) \ . \tag{9}$$

Here $\kappa^i = (\kappa^0, \boldsymbol{\kappa})$ and thus $\kappa_i = (\kappa_0, -\boldsymbol{\kappa})$ and

$$\kappa^0 = \kappa_0 = \sqrt{\kappa^2 + w/\tilde{\rho}} \ , \qquad (\kappa = |\boldsymbol{\kappa}|) \tag{10}$$

which follows from the identity $u^i u_i = 1$.

The special feature of a simple wave in any quasi-linear hyperbolic system of equations is that all quantities are considered as functions of only one variable which we call it the phase and denote by $\varphi = \varphi(\mathbf{r}, t)$. In our problem we write

$$\mathbf{U} = \mathbf{U}(\varphi) \ , \tag{11}$$

where

$$\mathbf{U} = (\tilde{\rho}, \kappa_1, \kappa_2, \kappa_3, S) \tag{12}$$

is the state vector of the system. Boillat[15, 17] showed that for a simple wave it is necessary to have

$$\frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \equiv \mathbf{n} = \mathbf{n}(\varphi) \qquad , \qquad -\frac{\partial\varphi/\partial t}{|\boldsymbol{\nabla}\varphi|} \equiv \lambda = \lambda(\varphi) \ . \tag{13}$$

In other words, the unit vector $\mathbf{n}$ normal to the wave front must be only a function of φ and the same is true for the phase velocity λ. Finally condition (13) implies that φ must satisfy[15, 17]

$$G(\varphi, \mathbf{r}, t) = f(\varphi) + \lambda(\varphi)t - \mathbf{r}\cdot\mathbf{n}(\varphi) \ , \tag{14}$$

in which f is an arbitrary differentiable function to be fixed through initial conditions. This equation clearly means that level surfaces of φ are flat planes. The functional form of $\mathbf{n}(\varphi)$ can not be determined from any equation and so it remains arbitrary to be flexible to fit with a given condition.

There are two noticeable points about these multidimensional simple waves. The first is the wave breaking at which the time and spatial derivatives of φ and so all variables diverge when $F \longrightarrow 0$ provided that

$$\frac{\partial\varphi}{\partial t} = -\frac{\lambda(\varphi)}{F} \qquad , \qquad \boldsymbol{\nabla}\varphi = \frac{\mathbf{n}(\varphi)}{F} \ , \tag{15}$$

$$F \equiv \frac{\partial G}{\partial\varphi} = \frac{df(\varphi)}{d\varphi} + \frac{d\lambda(\varphi)}{d\varphi}t - \mathbf{r}\cdot\frac{d\mathbf{n}(\varphi)}{d\varphi} = \frac{1}{|\boldsymbol{\nabla}\varphi|} \ . \tag{16}$$

Equations (15) are easily derived by implicit time and space differentiations of (14). Thus our simple wave solution is valid only when $F > 0$ and at any time and point where $F = 0$ the solution is not correct. The second point arises from the dependence of $\mathbf{n}$ on φ which implies that for two different values φ_1 and φ_2 of φ generally $\mathbf{n}(\varphi_1)$ and $\mathbf{n}(\varphi_2)$ are not parallel and thus they have an intersection on a line at which the solution is multi-valued which is not accepted. Hence, the domain of the valid solution must not contain such intersections. Both of these points demonstrate the “local” character of simple waves.

For a unidirectional 1-D simple wave where $\mathbf{n}$ is a constant vector it is possible for each value of φ to calculate the time of wave breaking ($F = 0$) as $t_c(\varphi) = -(df/d\varphi)/d\lambda/d\varphi$ and the earliest time of the wave breaking is obtained by solving the equation $(dt_c/d\varphi) = 0$[7]. Unfortunately such a nice situation does not hold in the multidimensional case when $\mathbf{n} = \mathbf{n}(\varphi)$. Let us see this

in a quantitative way. Singular points (wave breaking) not only must satisfy the simple wave condition (14) but also they should fulfil

$$F = 0 \ . \tag{17}$$

Thus, the wave breaking occurs on the line of intersection of the two perpendicular planes $G = 0$ and $F = 0$. This line is exactly the rotation axis of the wave front at φ when φ has an infinitesimal growth to $\varphi + \delta\varphi$. This will be easily seen if we observe that the wave front for $\varphi + \delta\varphi$ must satisfy

$$G(\varphi + \delta\varphi, \mathbf{r}, t) = 0 \ , \qquad \texttt{or} \qquad G(\varphi, \mathbf{r}, t) + F\delta\varphi = 0 \ ,$$

which again yields Eqs. (14) and (17). We may therefore conclude (without a rigorous proof) that if the wave breaking (singularity) line lies out of the region of the solution, the line of multi-valuedness will also lay in that region. Besides, since a line of singularity for each value of φ exists at each instant of time, it has no sense to speak about $t_c(\varphi)$. However if the fluid fills the whole space $\mathbb{R}^3$ we can obtain a moving surface constructed at any time exactly from all of these singular lines at that time. This surface is in fact the boundary between the forbidden and permitted regions relative to a simple wave solution.

Now we substitute the simple wave ansatz (11) into Eqs. (6)-(8) supplemented by Eq. (9) and then divide each equation by $\mid \boldsymbol{\nabla\varphi} \mid$ and use (13) to obtain the system of five quasi-linear coupled equations

$$A \ \frac{d\mathbf{U}}{d\varphi} = 0 \ , \tag{18}$$

where A is the 5×5 matrix with the following elements

$$A = \begin{bmatrix} \kappa_n - \frac{\lambda}{c}(\tilde{\rho}\frac{\partial\kappa_0}{\partial\tilde{\rho}} + \kappa_0) & \tilde{\rho}(n_1 - \frac{\lambda}{c}\frac{\kappa_1}{\kappa_0}) & \tilde{\rho}(n_2 - \frac{\lambda}{c}\frac{\kappa_2}{\kappa_0}) & \tilde{\rho}(n_3 - \frac{\lambda}{c}\frac{\kappa_3}{\kappa_0}) & -\frac{\lambda}{c}\tilde{\rho}\frac{\partial\kappa_0}{\partial S} \\ \frac{a^2 n_1}{\tilde{\rho}} & \kappa_n - \frac{\lambda}{c}\kappa_0 & 0 & 0 & \frac{P_S n_1}{\tilde{\rho}} \\ \frac{a^2 n_2}{\tilde{\rho}} & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 & 0 & \frac{P_S n_2}{\tilde{\rho}} \\ \frac{a^2 n_3}{\tilde{\rho}} & 0 & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 & \frac{P_S n_2}{\tilde{\rho}} \\ 0 & 0 & 0 & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 \end{bmatrix} . \tag{19}$$

In the above matrix we have used the following notations

$$\kappa_n \equiv \boldsymbol{\kappa} \cdot \mathbf{n} = \sum_{i=1}^{3} \kappa_i n_i \quad , \quad a^2 \equiv \left(\frac{\partial P}{\partial\tilde{\rho}}\right)_S \quad , \quad P_S \equiv \left(\frac{\partial P}{\partial S}\right)_{\tilde{\rho}} \ . \tag{20}$$

Moreover, in the calculation of $\frac{\partial\kappa_0}{\partial\tilde{\rho}}$ and $\frac{\partial\kappa_0}{\partial S}$ we must assume $w = w(\tilde{\rho}, S)$ and use Eq. (10) to express κ_0 explicitly as a function of all five variables $\mathbf{U} = (\tilde{\rho}, \boldsymbol{\kappa}, S)$.

Equation (18) has a nontrivial solution only when

$$\det(A) = 0 \ , \tag{21}$$

which constructs a fifth order equation for λ with a triple root

$$\lambda_1 = \lambda_2 = \lambda_3 = c\frac{\kappa_n}{\kappa_0} = v_n \ , \tag{22}$$

while the fourth and fifth roots λ_4 and λ_5 are the larger and smaller roots of the following quadratic equation respectively.

$$\left[\kappa_n - \frac{\lambda}{c}\left(\tilde{\rho}\frac{\partial\kappa_0}{\partial\tilde{\rho}} + \kappa_0\right)\right]\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right) = a^2\left(1 - \frac{\lambda}{c}\frac{\kappa_n}{\kappa_0}\right) \ . \tag{23}$$

The triplet root is the phase velocity for the two vortex modes and one entropy to be discussed in the next section. The roots λ_4 and λ_5 are the phase velocities for the forward and backward sound modes respectively. Although these modes have nonrelativistic analogue but they significantly differ from the nonrelativistic case.

Substitution of each value of the phase velocity into (18) yields some ordinary differential equations for $\mathbf{U}(\varphi)$ to be solved. For the entropy and vortex modes these equations are not difficult and some formal solutions both in the laboratory frame and the wave frame will be presented in Sections 3 and 4 respectively. Since the equations for the sound waves are complicated in the laboratory frame, we go to the wave frame but still they are difficult to solve and finally when we consider the physically common case of ultra-relativistic temperatures it will be possible to obtain some solutions in Sec. 4.

3 Vortex and entropy modes

If we substitute the triplet root $\lambda = c\frac{\kappa_n}{\kappa_0}$ into (18) we obtain

$$-\frac{\kappa_n}{\kappa_0}\frac{\partial\kappa_0}{\partial\tilde{\rho}}\frac{d\tilde{\rho}}{d\varphi} + \left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\cdot\frac{d\boldsymbol{\kappa}}{d\varphi} - \frac{\kappa_n}{\kappa_0}\frac{\partial\kappa_0}{\partial S}\frac{dS}{d\varphi} = 0\ , \tag{24}$$

$$a^2\frac{d\tilde{\rho}}{d\varphi} + P_S\frac{dS}{d\varphi} = \frac{dP}{d\varphi} = 0 \Longrightarrow P(\varphi) = \texttt{const}\ , \tag{25}$$

$$0\cdot\frac{dS}{d\varphi} = 0\ . \tag{26}$$

In Eq. (25) we have used the second and third equations of (20) together with (9). Equation (26) admits the two cases of constant entropy (vortex modes) and variable entropy (entropy mode).

3.1 Vortex modes

We have $dS = 0$ or

$$S(\varphi) = \text{const}\ , \tag{27}$$

which together with (25) and the state equation (9) yields the constancy of $\tilde{\rho}$ and so all thermodynamical variables. Thus, only the fluid velocity considered in $\boldsymbol{\kappa}$ and $\mathbf{n}$ change with φ which $\mathbf{n}(\varphi)$ is an arbitrary suitable function. Regarding the above results in Eqs. (24) and (10) one can obtain the equation for $\boldsymbol{\kappa}(\varphi)$.

$$\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\cdot\frac{d\boldsymbol{\kappa}}{d\varphi} = 0\ , \tag{28}$$

which must be supplemented by

$$\kappa_0 = \sqrt{\kappa^2 + w_0/\tilde{\rho}_0}\ , \tag{29}$$

where w_0 and $\tilde{\rho}_0$ are constant throughout the wave. The factor $\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)$ in (28) can not be zero because if it is zero we can make its inner product with $\mathbf{n}$ and obtain $\kappa_n^2 = \kappa^2 = \kappa_0^2$ which is impossible by (29). Therefore Eq. (28) is equivalent to

$$\frac{d\boldsymbol{\kappa}}{d\varphi} = \mathbf{X}(\varphi)\times\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\ , \tag{30}$$

where $\mathbf{X}(\varphi)$ is an arbitrary continuous function. It is possible to choose two functions $\mathbf{X}_1(\varphi)$ and $\mathbf{X}_2(\varphi)$ where $\mathbf{X}_1(\varphi)\cdot\mathbf{X}_2(\varphi)=0$ which gives two perpendicular and independent vortex modes similar to the nonrelativistic case[17].

It is also worth noting that we can define a generalized vortex

$$\mathbf{\Omega}\equiv\mathbf{\nabla}\times\boldsymbol{\kappa}=\mathbf{\nabla}\varphi\times\frac{d\boldsymbol{\kappa}}{d\varphi}=|\mathbf{\nabla}\varphi|\mathbf{n}\times\frac{d\boldsymbol{\kappa}}{d\varphi}\ , \tag{31}$$

which is constant not only on the wave front but also in advection with the fluid velocity:

$$\frac{\partial\mathbf{\Omega}}{\partial t}+(\mathbf{v}\cdot\mathbf{\nabla})\mathbf{\Omega}=|\mathbf{\nabla}\varphi|\left(\frac{c\kappa_n}{\kappa_0}-\lambda\right)\frac{d\mathbf{\Omega}}{d\varphi}=0\ , \tag{32}$$

This fact is consistent with the “frozen in” condition of filed lines of $\mathbf{\Omega}$[31]

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{\Omega}}{\gamma n}\right)+(\mathbf{v}\cdot\mathbf{\nabla})\left(\frac{\mathbf{\Omega}}{\gamma n}\right)=\left(\frac{\mathbf{\Omega}}{\gamma n}\cdot\mathbf{\nabla}\right)\mathbf{v}\ , \tag{33}$$

in which the number density in the laboratory frame γn is constant because

$$\mathbf{\nabla}\cdot\mathbf{v}=c\mathbf{\nabla}\cdot\left(\frac{\boldsymbol{\kappa}}{\kappa_0}\right)=c\frac{|\mathbf{\nabla}\varphi|}{\kappa_0}\left(\mathbf{n}-\frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\cdot\frac{d\boldsymbol{\kappa}}{d\varphi}=0\ , \tag{34}$$

according to Eq. (28). Finally by (31) we find

$$\mathbf{\Omega}\cdot\mathbf{\nabla}=|\mathbf{\nabla}\varphi|\mathbf{\Omega}\cdot\mathbf{n}\frac{d}{d\varphi}=0\ ,$$

by which Eq. (33) reduces to (32).

Equation (30) has many solutions since $\mathbf{X}(\varphi)$ is an arbitrary continuous function. It is therefore easy to choose some suitable simple forms for $\mathbf{X}$ such that Eq. (30) can be easily solved. As an example consider two different forms

$$\mathbf{X}=\alpha\boldsymbol{\kappa}\ ,\qquad\text{or alternatively}\qquad\mathbf{X}=\alpha\frac{\kappa_0^2}{\kappa_n}\mathbf{n}\ , \tag{35}$$

where α is a dimensionless constant. Selecting each value form for $\mathbf{X}$ from (35)causes to reduce Eq. (30) to

$$\frac{d\boldsymbol{\kappa}}{d\varphi}=\alpha\boldsymbol{\kappa}\times\mathbf{n}(\varphi)\ . \tag{36}$$

Regardless of the functional form of $\mathbf{n}(\varphi)$ it is obvious from (36) that $|\boldsymbol{\kappa}(\varphi)|$ is constant and from (29) κ_0 is also constant and only the direction of $\boldsymbol{\kappa}(\varphi)$ changes by φ. To obtain a more special solution let us consider a 2D simple wave with[17]

$$\mathbf{n}(\varphi)=(-\sin\varphi,\cos\varphi,0)\ , \tag{37}$$

in a Cartesian coordinate system. In this case we have

$$\boldsymbol{\kappa}=\kappa_n\mathbf{n}+\kappa_t\mathbf{t}+\kappa_3\mathbf{z}\ , \tag{38}$$

where

$$\kappa_n=\boldsymbol{\kappa}\cdot\mathbf{n}=-\kappa_1\sin\varphi+\kappa_2\cos\varphi\qquad,\qquad\kappa_t=\boldsymbol{\kappa}\cdot\mathbf{t}=-(\kappa_1\cos\varphi+\kappa_2\sin\varphi)\ , \tag{39}$$

in which $\mathbf{t} = (-\cos\varphi, -\sin\varphi, 0)$ is normal to $\mathbf{n}$. Substitution of (38) into (36) and using the relations

$$\frac{d\mathbf{n}}{d\varphi} = \mathbf{t} \quad , \quad \frac{d\mathbf{t}}{d\varphi} = -\mathbf{n} \quad , \quad \mathbf{n} \times \mathbf{t} = \mathbf{z} \; , \tag{40}$$

one finds

$$\frac{d\kappa_n}{d\varphi} = \kappa_t \; , \tag{41}$$

$$\frac{d\kappa_t}{d\varphi} + \kappa_n - \alpha\kappa_3 = 0 \; , \tag{42}$$

$$\frac{d\kappa_3}{d\varphi} = -\alpha\kappa_t \; . \tag{43}$$

Equations (41) and (43) yield

$$\kappa_3 = -\alpha\kappa_n + \bar{\kappa} \; , \tag{44}$$

where $\bar{\kappa}$ is a constant with the dimension of κ (velocity). Then we substitute κ_t from (41) and κ_3 from (44) into (42) to obtain

$$\frac{d^2\kappa_n}{d\varphi^2} + (1+\alpha^2)\kappa_n - \alpha\bar{\kappa} = 0 \; , \tag{45}$$

with a general solution

$$\kappa_n = \bar{\kappa}_n \cos[\sqrt{1+\alpha^2}(\varphi+\beta)] + \frac{\alpha}{1+\alpha^2}\bar{\kappa} \; , \tag{46}$$

where $\bar{\kappa}_n$ is a constant with the dimension of κ (velocity) while β is a dimensionless constant. Then from (41) we have

$$\kappa_t = -\sqrt{1+\alpha^2}\bar{\kappa}_n \sin[\sqrt{1+\alpha^2}(\varphi+\beta)] \; , \tag{47}$$

and from (44) we find

$$\kappa_3 = -\alpha\bar{\kappa}_n \cos[\sqrt{1+\alpha^2}(\varphi+\beta)] + \frac{1}{1+\alpha^2}\bar{\kappa} \; , \tag{48}$$

It is then straightforward to find κ_1 and κ_2 by the use of

$$\kappa_1 = -(\kappa_n \sin\varphi + \kappa_t \cos\varphi) \qquad , \qquad \kappa_2 = \kappa_n \cos\varphi - \kappa_t \sin\varphi \; . \tag{49}$$

As mentioned before a complete solution needs more detailed informations about initial and boundary conditions which is not of our interest here.

3.2 Entropy modes

Here we have $dS \neq 0$ in (26) and thus $\tilde{\rho}$ is not constant although by (25) P is still constant. It is possible to reduce Eq. (24) to

$$\mathbf{n} \cdot \left(\frac{d\boldsymbol{\kappa}}{d\varphi} - \frac{d(\ln\kappa_0)}{d\varphi}\boldsymbol{\kappa} \right) = 0 \; , \tag{50}$$

which gives

$$\frac{d\boldsymbol{\kappa}}{d\varphi} = \mathbf{Y}(\varphi) \times \mathbf{n} + \frac{d(\ln\kappa_0)}{d\varphi}\boldsymbol{\kappa} \; , \tag{51}$$

where $\mathbf{Y}(\varphi)$ is again an arbitrary continuous function. Let us again choose a suitable form for $\mathbf{Y}(\varphi)$ to simplify the solution. For example if $\mathbf{Y}$ is parallel to $\mathbf{n}$ we see from (51) that

$$\boldsymbol{\kappa} = \kappa_0 \mathbf{C} \ , \tag{52}$$

where $\mathbf{C} = (c_1, c_2, c_3)$ is a dimensionless constant vector. Since P is constant, the enthalpy w becomes only a function of $\tilde{\rho}$ and thus Eqs. (52) and (10) yield

$$\boldsymbol{\kappa} = \frac{\mathbf{C}}{\sqrt{1 - |\mathbf{C}|^2}} \sqrt{\frac{w(\tilde{\rho})}{\tilde{\rho}}} \ , \tag{53}$$

which is valid if $|\mathbf{C}| < 1$. It remains to specify the dependence of $\tilde{\rho}$ on φ. This dependence is arbitrary because the fixing of φ is under our control and we can assume it as an arbitrary function of one or more physical variables[17].

4 Simple waves presented in the wave frame

Sometimes apparent forms of those equations concerning a simple wave solution are so complicated and not soluble easily. A mathematical trick here is to rewrite all equations in terms of physical variables as measured in the wave frame. A wave frame depends on the special value of the phase φ that is, for any value of φ there is a plane wave front defined by Eq. (14) moving with the phase velocity $\mathbf{V}_{ph} = \lambda(\varphi)\mathbf{n}(\varphi)$ and we consider a Lorentz transformation from the laboratory frame to the frame co-moving with this wave front. It is thus clear that to each value of φ corresponds a unique wave frame.

We denote all quantities in the wave frame by a prime except scalar quantities such as $\tilde{\rho}, w, n$ etc which are either Lorentz invariant or defined in the proper frame co-moving with the fluid. Therefore for the 4-vector κ^i we have

$$\begin{aligned} \kappa_0 &= \cosh\xi \kappa'_0 + \sinh\xi \kappa'_n \ , \\ \kappa_n &= \cosh\xi \kappa'_n + \sinh\xi \kappa'_0 \ , \\ \boldsymbol{\kappa}_\perp &= \boldsymbol{\kappa}'_\perp \ , \end{aligned} \tag{54}$$

in which $\kappa'_n = \boldsymbol{\kappa}' \cdot \mathbf{n}$, $\boldsymbol{\kappa}'_\perp = \boldsymbol{\kappa}' - \kappa'_n \mathbf{n}$ and ξ depends on φ thorough

$$\tanh\xi = \frac{\lambda(\varphi)}{c} \ . \tag{55}$$

In the following subsections we apply this method for the vortex, entropy and sound modes and give simple formal solutions for each one.

4.1 Vortex modes in the wave frame

For this mode we already had $\frac{\lambda}{c} = \frac{\kappa_n}{\kappa_0}$ which by the substitution from (54) and (55) into it we find

$$\kappa'_n = 0 \ . \tag{56}$$

It is therefore seen that in the wave frame the phase velocity takes the simple form through the above equation. Then we substitute (54) and (55) into (28) and use (56) and the identity $\boldsymbol{\kappa}'_\perp \cdot \mathbf{n} = 0$ to obtain

$$\cosh^2\xi \mathbf{n} \cdot \frac{d\boldsymbol{\kappa}'_\perp}{d\varphi} - \frac{\sinh\xi}{\kappa'_0} \boldsymbol{\kappa}'_\perp \cdot \frac{d\boldsymbol{\kappa}'_\perp}{d\varphi} + \sinh\xi \frac{d\kappa'_0}{d\varphi} + \kappa'_0 \cosh\xi \frac{d\xi}{d\varphi} = 0 \ . \tag{57}$$

Since $(\kappa_0, \boldsymbol{\kappa})$ is a 4-vector, Eq. (29) is invariant and due to (56)we have

$$\kappa'_0 = \sqrt{\kappa'^2_\perp + w_0/\tilde{\rho}_0} \ , \tag{58}$$

by which Eq. (57) reduces to

$$\boldsymbol{\kappa}'_\perp \cdot \frac{d\mathbf{n}}{d\varphi} = -\mathbf{n} \cdot \frac{d\boldsymbol{\kappa}'_\perp}{d\varphi} = \frac{\kappa'_0}{\cosh \xi} \frac{d\xi}{d\varphi} \ . \tag{59}$$

Assuming $\mathbf{n}(\varphi)$ is a known function, $\boldsymbol{\kappa}'_\perp$ and ξ should satisfy Eq. (59) which obviously admits a large amount of freedom. As a very simple solution let us assume the restriction

$$\frac{d\boldsymbol{\kappa}'_\perp}{d\varphi} = -\kappa'_\perp \mathbf{n} \ , \tag{60}$$

which says that $\kappa'_\perp$ and κ'_0 are both constant and thus Eq. (59) can be easily solved to give

$$\int_{\xi_0}^{\xi} \frac{d\xi'}{\cosh \xi'} = \arctan(\sinh \xi) - \arctan(\sinh \xi_0) = \frac{\kappa'_\perp}{\kappa'_0} (\varphi - \varphi_0) \ . \tag{61}$$

From the above solution one can find ξ in terms of φ by which through Eq. (55) we have $\lambda(\varphi)$. For $\boldsymbol{\kappa}'_\perp$ we formally solve Eq. (60):

$$\boldsymbol{\kappa}'_\perp = -\kappa'_\perp \int_{\varphi_0}^{\varphi} \mathbf{n}(\varphi') d\varphi' + \boldsymbol{\kappa}'_\perp(\varphi_0) \ . \tag{62}$$

Here we gave a very restricted solution just as an example to show the procedure of the solution. Depending on initial and boundary conditions it is in principle possible to find more realistic solutions although it seems to be very difficult.

4.2 Entropy mode in the wave frame

Since the phase velocity $\frac{\lambda}{c} = \frac{\kappa_n}{\kappa_0}$ is the same as for the vortex mode, Eq. (56) is again valid here and transforming Eq. (50) in a manner similar to that performed for the vortex mode we again obtain Eq. (59) for the entropy wave too but here since P is constant, w is only a function of $\tilde{\rho}$ and thus

$$\kappa'_0 = \sqrt{\kappa'^2_\perp + w(\tilde{\rho})/\tilde{\rho}} \ . \tag{63}$$

Hence, the meaning of (59) for the entropy mode is different from this equation for the vortex mode. Again as a very restricted simple solution we suggest Eqs. (60) and (62) for $\boldsymbol{\kappa}'_\perp(\varphi)$ resulting in the constancy of $\kappa'_\perp$ and assume a given form for $\tilde{\rho}(\varphi)$ by which from (63) we have $\kappa'_0(\varphi)$ as a known function of φ and thus Eq. (59) has the formal solution

$$\arctan(\sinh \xi) - \arctan(\sinh \xi_0) = \kappa'_\perp \int_{\varphi_0}^{\varphi} \frac{d\varphi'}{\kappa'_0(\varphi')} \ . \tag{64}$$

4.3 Sound mode in the wave frame

At first we see that in the laboratory reference frame there are five equations included in (18) but due to Eq. (21) we have only four independent equations. The last (fifth) equation of (18)

by the use of (19) implies Eq. (27) also valid for the sound mode which its substitution into the first four equations of (18) yields

$$\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right)\frac{d\tilde{\rho}}{d\varphi} - \frac{\lambda}{c}\tilde{\rho}\frac{d\kappa_0}{d\varphi} + \tilde{\rho}\mathbf{n}\cdot\frac{d\boldsymbol{\kappa}}{d\varphi} = 0 \ , \tag{65}$$

as the continuity equation and

$$\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right)\frac{d\boldsymbol{\kappa}}{d\varphi} + \frac{a^2}{\tilde{\rho}}\frac{d\tilde{\rho}}{d\varphi}\mathbf{n} = 0 \ , \tag{66}$$

as the momentum equation in which a^2 is defined from (20). Equations (65) and (66) are four equations but only three of them are independent while the phase velocity is $\lambda = \lambda_4$ or $\lambda = \lambda_5$ which are the roots of the quadratic equation (23).

Let us rewrite Eqs. (65) and (66) in terms of the wave frame quantities through Eqs. (54) to find

$$\kappa'_n\frac{d\tilde{\rho}}{d\varphi} + \tilde{\rho}\kappa'_0\frac{d\xi}{d\varphi} + \tilde{\rho}\frac{d\kappa'_n}{d\varphi} + \cosh\xi\ \tilde{\rho}\mathbf{n}\cdot\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} = 0 \ , \tag{67}$$

and

$$\cosh\xi\ \frac{a^2}{\tilde{\rho}\kappa'_n}\frac{d\tilde{\rho}}{d\varphi}\mathbf{n} + \frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} + \mathbf{n}\frac{d}{d\varphi}(\kappa'_n\cosh\xi + \kappa'_0\sinh\xi) + (\kappa'_n\cosh\xi + \kappa'_0\sinh\xi)\frac{d\mathbf{n}}{d\varphi} = 0 \ , \tag{68}$$

with only three independent equations.

It is also necessary to rewrite the quadratic equation (23) (whose roots are the sound waves λ_4 and λ_5) in terms of the wave frame quantities. Substitution of (54) and (55) into (23) yields

$$\left(\kappa'_n - \tilde{\rho}\frac{\partial\kappa_0}{\partial\tilde{\rho}}\sinh\xi\right)\kappa'_n = a^2\frac{\kappa'_0}{\kappa'_0 + \kappa'_n\tanh\xi} \ . \tag{69}$$

Now we notice that according to Eq. (10) we have

$$\frac{\partial\kappa_0}{\partial\tilde{\rho}} = \frac{1}{2\kappa_0}\frac{\partial}{\partial\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right)_S = \frac{1}{2}\frac{1}{\kappa'_0\cosh\xi + \kappa'_n\sinh\xi}\frac{d}{d\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right) \ , \tag{70}$$

where we have used Eq. (27) by which the entropy is constant and thus w is only a function of $\tilde{\rho}$. We then substitute Eq. (70) into (69) to find

$$(\kappa'^2_n - a^2)\kappa'_0 + \left[\kappa'^2_n - \frac{\tilde{\rho}}{2}\frac{\partial}{\partial\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right)_S\right]\kappa'_n\tanh\xi = 0 \ , \tag{71}$$

in which

$$\kappa'_0 = \sqrt{\kappa'^2 + w/\tilde{\rho}} \ . \tag{72}$$

Equation (71) determines the phase velocity $\tanh\xi$ in terms of the physical variables measured in the wave frame but since it is generally complicated depending on the explicit form of $w(\tilde{\rho})$ we can not go further. However, it is possible to continue for the ultra-relativistic case when $K_BT \gg mc^2$ which implies that

$$\frac{w}{w_\circ} = \left(\frac{T}{T_\circ}\right)^4 \quad , \quad \frac{n}{n_\circ} = \left(\frac{T}{T_\circ}\right)^3 \quad , \quad P = \frac{1}{4}w \ ,$$

by which it is easy to see

$$\frac{w}{\tilde{\rho}} = \frac{w_\circ}{\tilde{\rho}^2_\circ}\tilde{\rho} \quad , \quad a^2 = \frac{dP}{d\tilde{\rho}} = \frac{1}{2}\frac{w_\circ}{\tilde{\rho}^2_\circ}\tilde{\rho} \ , \tag{73}$$

where the subscript "$\circ$" denotes the equilibrium point of the fluid at which it is at rest. By the above simplifications Eq. (71) reduces to

$$(\kappa_n'^2 - \frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho})(\kappa_0' + \kappa_n' \tanh\xi) = 0 \ .$$

Since $|\kappa_0'/\kappa_n'| > 1$ while $|\tanh\xi| < 1$, the second factor in the above equation can not be zero and thus for the sound mode in the ultra-relativistic case we obtain

$$\kappa_n' = \mp a = \mp\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}} \qquad , \qquad \kappa_0' = \sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}} \ . \tag{74}$$

Here the upper (minus) sign indicates the case where the fluid velocity is negative with respect to the wave front which means that the wave runs faster than the fluid and thus it refers to the forward sound wave. Similarly the lower (plus) sign refers to the backward sound wave.

As mentioned before there are only three independent equations namely Eq. (68) when Eq. (74) is substituted into it. It is more convenient to to write Eq. (68) in the three orthogonal directions $\mathbf{n}$, $\boldsymbol{\kappa'}_\perp$ and $d\mathbf{n}/d\varphi$. Thus, making the scalar product of (68)[after the substitution of (74) into it] by $\mathbf{n}$ yields

$$\left(\sinh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}} \mp \cosh\xi\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}}\right)\left(\frac{3}{2}\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}}\frac{1}{\sqrt{\tilde{\rho}(\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho})}}\frac{d\tilde{\rho}}{d\varphi} \mp \frac{d\xi}{d\varphi}\right)$$

$$= -\frac{\sinh\xi}{\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}}}\boldsymbol{\kappa'}_\perp \cdot \frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} + \boldsymbol{\kappa'}_\perp \cdot \frac{d\mathbf{n}}{d\varphi} \ , \tag{75}$$

in which we have used the identity $\boldsymbol{\kappa'}_\perp \cdot \mathbf{n} = 0$ which gives

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} \cdot \mathbf{n} + \boldsymbol{\kappa'}_\perp \cdot \frac{d\mathbf{n}}{d\varphi} = 0 \ . \tag{76}$$

Next, let us make the scalar product of (68) by $\boldsymbol{\kappa'}_\perp$:

$$\boldsymbol{\kappa'}_\perp \cdot \frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} = -\left(\sinh\xi\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}} \mp \cosh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}}\right)\boldsymbol{\kappa'}_\perp \cdot \frac{d\mathbf{n}}{d\varphi} \ . \tag{77}$$

Finally the scalar product of (68) by $d\mathbf{n}/d\varphi$ is

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} \cdot \frac{d\mathbf{n}}{d\varphi} = -\left(\sinh\xi\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}} \mp \cosh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}}\right)|d\mathbf{n}/d\varphi|^2 \ . \tag{78}$$

Thus we should solve the system of equations (75), (77) and (78) provided that $\mathbf{n}(\varphi)$ is a known function.

If non of Eqs. (77) and (78) vanishes one can divide them by each other and after some simple vector calculations obtain

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} \cdot \left[\frac{d\mathbf{n}}{d\varphi} \times \left(\frac{d\mathbf{n}}{d\varphi} \times \boldsymbol{\kappa'}_\perp\right)\right] = 0 \ , \tag{79}$$

which gives

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} = \mathbf{Z}(\varphi) \times \left[\frac{d\mathbf{n}}{d\varphi} \times \left(\frac{d\mathbf{n}}{d\varphi} \times \boldsymbol{\kappa'}_\perp\right)\right] \ , \tag{80}$$

where $\mathbf{Z}(\varphi)$ is an arbitrary continuous function. We will not go further in this way but alternatively seek fore more simple solutions. If we assume $\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} = 0$ then it is possible to show after some calculations that this is not a consistent solution for the system of equations (75), (77) and (78). However, a consistent simple solution is found under the assumption

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} \cdot \mathbf{n} = -\boldsymbol{\kappa'}_\perp \cdot \frac{d\mathbf{n}}{d\varphi} = 0 \ . \tag{81}$$

This condition with the help of (77) gives

$$\kappa'_\perp = \texttt{const} \equiv \bar{\kappa}'_\perp \ . \tag{82}$$

By the use of (81) and (82) since $|\tanh\xi| < 1$ we find a differential equation relating $\tilde{\rho}$ to ξ whose solution is

$$\tilde{\rho} = \frac{\bar{\kappa}'^2_\perp \tilde{\rho}^2_\circ}{3w_\circ} \left\{ \cosh\left[\pm\frac{2}{\sqrt{3}}(\xi - \xi_\circ) + \cosh^{-1}\left(1 + \frac{3w_\circ}{\bar{\kappa}'^2_\perp \tilde{\rho}_\circ}\right)\right] - 1 \right\}, \tag{83}$$

where the upper (positive) sign refers to the forward and the lower (negative) sign denotes the backward sound wave. Now we should find $\boldsymbol{\kappa'}_\perp$. It is clear from (81) and (82) that $\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi}$ is perpendicular to both $\boldsymbol{\kappa'}_\perp$ and $\mathbf{n}$ thus $\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi}$ is parallel to $\mathbf{n} \times \boldsymbol{\kappa'}_\perp$. On the other hand the identity $\mathbf{n} \cdot \frac{d\mathbf{n}}{d\varphi} = 0$ and Eq. (81) imply that $\frac{d\mathbf{n}}{d\varphi}$ is also parallel to $\mathbf{n} \times \boldsymbol{\kappa'}_\perp$. Therefore we conclude that

$$\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi} = \pi(\varphi)\frac{d\mathbf{n}}{d\varphi} \ ,$$

or equivalently

$$\boldsymbol{\kappa'}_\perp(\varphi) = \int_{\varphi_\circ}^{\varphi} \pi(\varphi')\frac{d\mathbf{n}(\varphi')}{d\varphi'} d\varphi' + \boldsymbol{\kappa'}_\perp(\varphi_\circ) \ , \tag{84}$$

where $\pi(\varphi')$ is an arbitrary nonzero continuous scalar function (we remember that $\frac{d\boldsymbol{\kappa'}_\perp}{d\varphi}$ can not vanish). Finally we substitute (84) and (82) into (78) to obtain

$$\left(\sinh\xi \sqrt{\kappa'^2_\perp + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}^2_\circ}\tilde{\rho}} \mp \cosh\xi \sqrt{\frac{w_\circ}{2\tilde{\rho}^2_\circ}\tilde{\rho}} \right) = -\pi(\varphi) \ . \tag{85}$$

Equations (83) and (85) are used to express both $\tilde{\rho}$ and ξ as functions of φ and this means that the problem is formally solved. Substituting of all physical quantities obtained above into the Lorentz transformation (54) provides all the things in the laboratory frame.

5 Symmetry analysis for the vortex mode equation

Before starting this section, let us mention that from here on we change all previous notations to quite new applications. So we forget the meaning of all letters or symbols used in all preceding sections and introduce new applications of them.

We consider Eq. (28) as a first order ODE and rewrite it in the following form

$$\frac{d\mathbf{k}}{dt} \cdot \left(\mathbf{n} - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w}\mathbf{k}\right) = 0, \tag{86}$$

where w is a constant, t is treated as the wave phase, and $\mathbf{k} = (k_1, k_2, k_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$ are some vectors in $\mathbb{R}^3$ having the physical meaning of $\boldsymbol{\kappa}$ and unit normal vector to the wave front

resp. We concern with the latter equation to find its point and contact symmetry properties and also give its fundamental invariants and a form of general solutions.

It is notable here that in the mathematical structure of the simple wave solution, it is necessary to take the unit length for $\mathbf{n}$ (see Eq. (13)). However since Eq. (86) is homogeneous and linear with respect to $\mathbf{n}$, this condition is not essential in obtaining any solution. This condition appears important only for the compatibility of the simple wave structure. Regarding this fact, we make our symmetry analysis in both cases of arbitrary length and unit length for $\mathbf{n}$ and compare the results with each other.

Throughout this section we assume that indices i, j varies between 1 and 3. Also each index of a function implies the derivation of the function with respect to it, unless specially stated otherwise.

5.1 The point Symmetry of the Equation

To find the symmetry group of Eq. (86) by Lie infinitesimal method, we follow the method presented in [32]. We find infinitesimal generators as well as the Lie algebra structure of the symmetry group of that equation. In this subsection, we are concerned with the action of the point transformation group.

The equation is a relation along with the variables of 1–jet space $J^1(\mathbb{R},\mathbb{R}^6)$ with (local) coordinate $(t,\mathbf{k},\mathbf{n},\mathbf{q},\mathbf{p}) = (t,k_i,n_j,q_r,p_s)$ (for $1 \leq i,j,r,s \leq 3$), where this coordinate involving an independent variable t and 6 dependent variables k_i, n_j and their first derivatives q_r, p_s with respect to t resp.

Let $\mathcal{M}$ be the total space of independent and dependent variables. The solution space of Eq. (86), (if it exists) is a subvariety $S_\Delta \subset J^1(\mathbb{R},\mathbb{R}^6)$ of the first order jet bundle of one–dimensional submanifolds of $\mathcal{M}$, that is, graph of functions k_i, n_j, of elements $(t, k_i(t), n_j(t))$ satisfying Eq. (86) and the relations $q_1 = \dfrac{\partial k_1}{\partial t}$, $q_2 = \dfrac{\partial k_2}{\partial t}$, $q_3 = \dfrac{\partial k_3}{\partial t}$, $p_1 = \dfrac{\partial n_1}{\partial t}$, $p_2 = \dfrac{\partial n_2}{\partial t}$, $p_3 = \dfrac{\partial n_3}{\partial t}$ are all fulfilled.

We define a point transformation on $\mathcal{M}$ with relations

$$\tilde{t} = \phi(t,k_i,n_j), \qquad \tilde{k}_r = \chi_r(t,k_i,n_j), \qquad \tilde{n}_s = \psi_s(t,k_i,n_j).$$

where ϕ, χ_r and ψ_s are arbitrary smooth functions. Let

$$v := T\,\frac{\partial}{\partial t} + \sum_{i=1}^{3}\Big(K_i\frac{\partial}{\partial k_i} + N_i\frac{\partial}{\partial n_i}\Big) \tag{87}$$

be the general form of infinitesimal generators that signify the Lie algebra $\mathfrak{g}$ of the symmetry group G of Eq. (86). In this relation, T, K_i and N_j are smooth functions of variables t, k_i and n_j. The first order prolongation [32] of v is as follows

$$v^{(1)} \;:=\; v + \sum_i K_i^t\frac{\partial}{\partial q_i} + \sum_j N_j^t\frac{\partial}{\partial p_j},$$

where $K_i^t = D_t\,Q_1^i + T\,q_{i,t}$ and $N_j^t = D_t\,Q_2^j + T\,p_{j,t}$, in which D_t is the total derivative and $Q_1^i = K_i - T\,q_i$ and $Q_2^j = N_j - T\,p_j$ are characteristics of vector field v [32]. By applying $v^{(1)}$ on (86), we obtain the following relation

$$\begin{aligned}\sum_i \Big\{\Big[K_i\Big((\mathbf{k}^2+w)(\mathbf{k}\cdot\mathbf{n}+n_i) - 2\,k_i\Big) + N_i(\mathbf{k}^2+w)(\mathbf{k}^2+w-k_i^2)\Big]\,q_i + \Big[K_{i\,t} - \sum_j (q_j\,K_{i\,k_j}\\ +p_j\,K_{i\,n_j}) - q_i\Big(T_t - \sum_j (q_j\,T_{k_j} + p_j\,T_{n_j})\Big)\Big]\Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})\Big)\Big\} = 0,\end{aligned} \tag{88}$$

whenever Eq. (86) is satisfied. We may prescribe t, k_i, n_j, q_r, p_s $(1 \le i, j, r, s \le 3)$ arbitrarily while functions T, K_i and N_j only depend on t, k_i, n_j. Thus Eq. (88) will be satisfied if and only if we have the following equations:

$$\sum_i \Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})\Big)\, K_{i\,t} = 0, \tag{89}$$

$$\begin{aligned} & N_i(\mathbf{k}^2+w)(\mathbf{k}^2+w-k_i^2) - \sum_j \Big(n_j(\mathbf{k}^2+w) - k_j(\mathbf{k}\cdot\mathbf{n})\Big)\, K_{j\,k_i} \\ & -\Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})\Big)\, T_t + K_i\Big((\mathbf{k}^2+w)(\mathbf{k}\cdot\mathbf{n}+n_i) - 2\,k_i\Big) = 0, \end{aligned} \tag{90}$$

$$T_{k_i}\Big(n_j(\mathbf{k}^2+w) - k_j(\mathbf{k}\cdot\mathbf{n}\Big) = 0, \tag{91}$$

$$T_{k_i}\Big(n_j(\mathbf{k}^2+w) - k_j(\mathbf{k}\cdot\mathbf{n}\Big) + T_{k_j}\Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n}\Big) = 0, \tag{92}$$

$$\sum_j \Big(n_j(\mathbf{k}^2+w) - k_j(\mathbf{k}\cdot\mathbf{n}\Big)\, K_{j\,n_i} = 0. \tag{93}$$

These equations are called the determining equations. From Eqs. (90) for each i we have

$$\begin{aligned} N_i &= (\mathbf{k}^2+w-k_i^2)^{-1}\Big\{\Big(2\,k_i(\mathbf{k}^2+w) - (\mathbf{kn}+n_i)\Big)\, K_i + \sum_j \Big(n_j(\mathbf{k}^2+w) - k_j(\mathbf{k}\cdot\mathbf{n})\Big)\, K_{j\,k_i} \\ &\quad +\Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})\Big)\, T_t\Big\}. \end{aligned} \tag{94}$$

Since $\mathbf{n} \ne 0$, without loss of generality, one may assume that $n_1 \ne 0$. Also since $\mathbf{k}^2 + w \ne 0$ so by Eqs. (91) and (92) we conclude that T just depends on t:

$$T = T(t). \tag{95}$$

By solving Eq. (89) with respect to t, we deduce the following relation of K_i s

$$\sum_i \Big(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})\Big)\, K_i = 0. \tag{96}$$

After differentiating of the latter equation with respect to n_j when we apply Eqs. (93) we lead to the following relations

$$(\mathbf{k}^2+w-k_i^2)\, K_i - \sum_{j\ne i} k_i\, k_j\, K_j = 0.$$

These relations suggest the general forms of K_1, K_2 and K_3 as follows

$$K_1 = K_2 = K_3 = 0. \tag{97}$$

By applying (97) on relations (94) for different values of i, the forms of N_i s are also achieved:

$$N_i \;=\; (\mathbf{k}^2+w-k_i^2)^{-1}(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n}))\, T_t. \tag{98}$$

Finally, the general form of infinitesimal generators as elements of point symmetry algebra of Eq. (86), which we call them *point infinitesimal generators*, for arbitrary functions T is as follows

$$v = v_T := T\,\frac{\partial}{\partial t} + \sum_{i=1}^{3}\Big\{(\mathbf{k}^2+w-k_i^2)^{-1}(n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n}))\, T_t\Big\}\frac{\partial}{\partial n_i}. \tag{99}$$

The Lie bracket (commutator) of every two vector fields in the form of (99) straightforwardly is an infinitesimal operator in the same form of them. More explicitly, the commutator of operators v_T and $v_{\overline{T}}$ is vector field $v_{T\overline{T}_t - T_t\overline{T}}$. Hence, the Lie algebra $\mathfrak{g} = \langle v_T\rangle$ of point symmetry group G, when T is an arbitrary smooth function which depends on t, is a Lie algebra.

Theorem 1. *The set of all point infinitesimal generators in the form of (99) is an infinite–dimensional Lie algebra of equation (86) for arbitrary* $\mathbf{n}$ *(not necessarily unit).*

According to theorem 2.74 of [32], the invariants $u = I(t, k_1, k_2, k_3, n_1, n_2, n_3)$ of one–parameter group with infinitesimal generators in the form of (99) satisfy the linear homogeneous partial differential equations of first order:

$$v[I] = 0.$$

The solutions of the latter, are found by the method of characteristics (See [32] and [33] for details). So we can replace the above equation by the following characteristic system of ordinary differential equations

$$\frac{dt}{T} = \frac{dk_1}{K_1} = \frac{dk_2}{K_2} = \frac{dk_3}{K_3} = \frac{dn_1}{N_1} = \frac{dn_2}{N_2} = \frac{dn_3}{N_3}. \tag{100}$$

By solving Eqs. (100) of the differential generator (99), we (locally) find the following general solutions

$$\begin{aligned} &I_\alpha(t, \mathbf{k}, \mathbf{n}) = k_\alpha = d_\alpha, && \text{for } \alpha = 1, 2, 3 \\ &I_{\beta+3}(t, \mathbf{k}, \mathbf{n}) = \frac{1}{T}\Big\{n_\beta(\mathbf{k}^2 + w - k_\beta^2) - k_\beta(\mathbf{k}\cdot\mathbf{n})\Big\} = d_{\beta+3}, && \text{for } \beta = 1, 2, 3. \end{aligned} \tag{101}$$

where d_α and d_β are constants. The functions $I_1, I_2, \cdots, I_6$ form a complete set of functionally independent invariants of one–parameter group generated by (99) (see [32]).

Similar to the theorem of section 4.3.3 of [33], the derived invariants (101) as independent first integrals of the characteristic system of the infinitesimal generator (99) provide the general solution

$$S(t, \mathbf{k}, \mathbf{n}) := \mu(I_1(t, \mathbf{k}, \mathbf{n}), I_2(t, \mathbf{k}, \mathbf{n}), \cdots, I_6(t, \mathbf{k}, \mathbf{n})),$$

with an arbitrary function μ, which satisfies in the equation $v[\mu] = 0$. This theorem can be extended for each finite set of independent first integrals (invariants) of characteristic system provided with an infinitesimal generator.

In the following, we give some examples provided with different selections of coefficients of Eq. (99) to show the method explicitly. We assumed that each appeared coefficient of vector fields is nonzero.

Example 1. If we assume that $T = 1$, then the infinitesimal operator (99) reduces to the following vector field $v_1 = \frac{\partial}{\partial t}$ and the group transformations (or flows) for the parameter s are expressible as $(t, \mathbf{k}, \mathbf{n}) \to (t + s, \mathbf{k}, \mathbf{n})$ which form the (local) symmetry group of v_1.

The derived invariants in this case will be as follows

$$I_\alpha = k_\alpha, \qquad I_{\beta+3} = n_\beta(\mathbf{k}^2 + w - k_j^2) - k_\beta(\mathbf{k}\cdot\mathbf{n}), \qquad \text{for } \alpha, \beta = 1, 2, 3.$$

Therefore the general solution corresponding to v_1 when μ is an arbitrary function, will be

$$S(t, \mathbf{k}, \mathbf{n}) = \mu\Big(\mathbf{k}\,,\, \mathbf{n}(\mathbf{k}^2 + w - k_1^2) - \mathbf{k}(\mathbf{k}\cdot\mathbf{n})\Big).$$

Example 2. Let $T = t$, then the infinitesimal generator is

$$v_2 = t\,\frac{\partial}{\partial t} + \sum_{j=1}^{3} (\mathbf{k}^2 + w - k_i^2)^{-1}(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n}))\frac{\partial}{\partial n_j},$$

Then, the flows of v_2 for various values of parameter s are

$$(t, k_i, n_j) \to \Big(t\,e^s, k_i, (\mathbf{k}^2 + w - k_j^2)^{-1}\Big\{(n_j(\mathbf{k}^2 + w) - k_j(\mathbf{k}\cdot\mathbf{n}))\,e^s + (\mathbf{k}\cdot\mathbf{n} - n_j\,k_j)\,k_j\Big\}\Big).$$

Also, we have the below invariants

$$I_\alpha = k_\alpha, \qquad I_{\beta+3} = \frac{1}{t}\Big(n_\beta(\mathbf{k}^2 + w - k_j^2) - k_\beta(\mathbf{k}\cdot\mathbf{n})\Big), \qquad \text{for } \alpha, \beta = 1, 2, 3,$$

whenever defined and the general solution of Eq. (86) as

$$S(t, \mathbf{k}, \mathbf{n}) = \mu\Big(\mathbf{k}\,,\ \frac{1}{t}\Big(\mathbf{n}(\mathbf{k}^2 + w - k_j^2) - \mathbf{k}(\mathbf{k}\cdot\mathbf{n})\Big)\Big),$$

where μ is an arbitrary function.

Example 3. In the case $T = e^t$ the infinitesimal generator (99) changes to

$$v_3 = e^t\Big\{\frac{\partial}{\partial t} + \sum_{j=1}^{3} (\mathbf{k}^2 + w - k_i^2)^{-1}(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n}))\,\frac{\partial}{\partial n_j}\Big\},$$

with group transformations of the parameter s transforming (t, k_i, n_j) to

$$P(s) = \Big(\ln\{e^t(1 - s\,e^t)^{-1}\}\,,\ k_i\,,\ k_j\,s\,e^t\,(1 - s\,e^t)^{-1}\Big(n_j - (\mathbf{k}\cdot\mathbf{n} - n_j k_j)(\mathbf{k}^2 + w - k_j^2)^{-1}\Big)\Big)\,,$$

wherever defined. Independent invariants are

$$I_\alpha = k_\alpha, \qquad I_{\beta+3} = e^{-t}\Big(n_\beta(\mathbf{k}^2 + w - k_j^2) - k_\beta(\mathbf{k}\cdot\mathbf{n})\Big), \qquad \text{for } \alpha, \beta = 1, 2, 3,$$

and hence the general solution of (86) with respect to infinitesimal operator v_3 is an arbitrary function of these invariants. Indeed, if $u = f(t, k_i, n_j)$ be a solution of Eq. (86) then so is $u = f(P(s))$ for each s.

When we assume that $\mathbf{n}$ to be of unit length, then by the action of $v^{(1)}$ (the first prolongation of the general form (87) of infinitesimal generator v) on relation $n_1^2 + n_2^2 + n_3^2 = 1$ we tend to the following equation

$$n_1\,N_1 + n_2\,N_2 + n_3\,N_3 = 0.$$

The last equation along with the deduced form of N_i s in (98) imply that $T_t = 0$. Hence $T = c$ for arbitrary constant c and for each i, $N_i = 0$. Therefore, the form of infinitesimal generators reduces from relation (99) to the below expression

$$v = \frac{\partial}{\partial t}.$$

Theorem 2. *The point Lie algebra of equation (86) when $\mathbf{n}$ is a unit normal vector to the wave front is $\mathfrak{g} = \langle\frac{\partial}{\partial t}\rangle$ isomorphic to the Lie algebra $\mathbb{R}$. Therefore the point symmetry group of the equation with this additional condition is the group of phase translations.*

5.2 The Contact Symmetry of the Equation

In continuation, we change the group action and find symmetry group and invariants of Eq. (86) up to the contact transformation groups. According to Bäcklund theorem [32], if the number of dependent variables be greater than one (like our problem), then each contact transformation is the prolongation of a point transformation. In this subsection, we earn the structure of infinitesimal generators of contact transformations. The normal vector to wave front, **n**, is assumed to be either arbitrary (not necessarily unit) or of unit length and then we find the contact symmetry properties of the Eq. (86) in these two situations.

We suppose that the general form of a contact transformation be as following

$$\begin{aligned}
&\widetilde{t} = \phi(t, k_i, n_j, q_r, p_s), \qquad \widetilde{k}_l = \chi_l(t, k_i, n_j, q_r, p_s), \quad \widetilde{n}_m = \psi_m(t, k_i, n_j, q_r, p_s),\\
&\widetilde{q}_n = \eta_n(t, k_i, n_j, q_r, p_s), \quad \widetilde{p}_u = \zeta_u(t, k_i, n_j, q_r, p_s),
\end{aligned}$$

where i, j, l, m, n and u varies between 1 and 6; and $\phi, \chi_l, \psi_m, \eta_n$ and ζ_u are arbitrary smooth functions. In this case of group action, an infinitesimal generator which is a vector field in $J^1(\mathbb{R}, \mathbb{R}^6)$, has the following general form

$$v := T\frac{\partial}{\partial t} + \sum_{i=1}^{3}\{K_i\frac{\partial}{\partial k_i} + N_i\frac{\partial}{\partial n_i} + Q_i\frac{\partial}{\partial q_i} + P_i\frac{\partial}{\partial p_i}\}, \tag{102}$$

for arbitrary smooth functions T, K_l, N_m, Q_m, P_u ($l = 1, 2$ and $1 \leq m, n, u \leq 3$).

Since our computations are done in 1–jet space, so we do not need to lift v to higher jet spaces and hence we act v (itself) on the Eq. (86), then we find the following relation

$$\begin{aligned}
&\sum_{i=1}^{3}\Big\{N_i\, q_i(\mathbf{k}^2 + w)(\mathbf{k}^2 + w - k_i^2) + Q_i(\mathbf{k}^2 + w)(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n}))\\
&-K_i\, q_i(n_i(\mathbf{k}^2 + w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w - 2\,k_i))\Big\} = 0.
\end{aligned}$$

Since $\mathbf{n} \neq 0$, so without less of generality, we can suppose that $n_1 \neq 0$, then the solution to this equation for would be

$$\begin{aligned}
Q_1 \;=\;& (\mathbf{k}^2 + w)^{-1}(n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n}))^{-1}\Big\{\sum_{i=1}^{3}\Big(K_i\, q_i(n_i(\mathbf{k}^2 + w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w - 2\,k_i))\\
&-N_i\, q_i(\mathbf{k}^2 + w)(\mathbf{k}^2 + w - k_i^2)\Big) - \sum_{j=2}^{3} Q_i(\mathbf{k}^2 + w)(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n}))\Big\}.
\end{aligned}$$

Therefore, the infinitesimal generator which we call it as *contact infinitesimal generator* is in the following form

$$\begin{aligned}
v \;=\;& T\frac{\partial}{\partial t} + \sum_i K_i\Big(\frac{\partial}{\partial k_i} + \frac{n_i(\mathbf{k}^2 + w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w - 2\,k_i))}{(\mathbf{k}^2 + w)(n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n}))}\, q_i\,\frac{\partial}{\partial q_1}\Big)\\
&+\sum_i N_i\Big(\frac{\partial}{\partial n_i} - \frac{\mathbf{k}^2 + w - k_i^2))}{n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n})}\, q_i\,\frac{\partial}{\partial q_1}\Big) + \sum_i P_i\,\frac{\partial}{\partial p_i}\\
&+\sum_{j=2,3} Q_j\Big(\frac{\partial}{\partial q_j} - \frac{n_j(\mathbf{k}^2 + w) + k_j(\mathbf{k}\cdot\mathbf{n})}{n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n})}\,\frac{\partial}{\partial q_1}\Big).
\end{aligned} \tag{103}$$

Table 5.2

The commutators table provided by contact symmetry.

	v_T	v_{K_α}	v_{N_β}	v_{Q_γ}	v_{P_η}
v_T	0	$v_T + v_{K_\alpha}$	$v_T + v_{N_\beta}$	$v_T + v_{Q_\gamma}$	$v_T + v_{P_\eta}$
v_{K_α}	$-v_T - v_{K_\alpha}$	0	$v_{K_\alpha} + v_{N_\beta}$	$v_{K_\alpha} + v_{Q_\beta}$	$v_{K_\alpha} + v_{P_\eta}$
v_{N_β}	$-v_T - v_{N_\beta}$	$-v_{K_\alpha} - v_{N_\beta}$	0	$v_{N_\beta} + v_{Q_\gamma}$	$v_{N_\beta} + v_{P_\eta}$
v_{Q_γ}	$-v_T - v_{Q_\gamma}$	$-v_{K_\alpha} - v_{Q_\gamma}$	$-v_{N_\beta} - v_{Q_\gamma}$	0	$v_{Q_\gamma} + v_{P_\eta}$
v_{P_η}	$-v_T - v_{P_\eta}$	$-v_{K_\alpha} - v_{P_\eta}$	$-v_{N_\beta} - v_{Q_\eta}$	$-v_{Q_\gamma} - v_{P_\eta}$	0

One may divide the latter form to the following vector fields, to consist a basis for Lie algebra $\mathfrak{g} = \langle v \rangle$ of contact symmetry group G:

$$\begin{aligned}
&v_T = T\,\frac{\partial}{\partial t}, && v_{K_i} = K_i\Big(\frac{\partial}{\partial k_i} + \frac{n_i(\mathbf{k}^2+w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2+w-2\,k_i))}{(\mathbf{k}^2+w)(n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n}))}\,q_i\,\frac{\partial}{\partial q_1}\Big),\\
&v_{P_i} = P_i\,\frac{\partial}{\partial p_i}, && v_{N_i} = N_i\Big(\frac{\partial}{\partial n_i} - \frac{\mathbf{k}^2+w-k_i^2))}{n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n})}\,q_i\,\frac{\partial}{\partial q_1}\Big),\\
&v_{Q_j} = Q_j\Big(\frac{\partial}{\partial q_j} - \frac{n_j(\mathbf{k}^2+w) + k_j(\mathbf{k}\cdot\mathbf{n})}{n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n})}\,\frac{\partial}{\partial q_1}\Big),
\end{aligned} \tag{104}$$

where $1 \le i \le 3$ and $2 \le j \le 3$. The commutator of every two of vector fields (104) is a linear combination of two operators (104) which are generally in the form of those two operators again. Thus these vector fields construct a basis for the Lie algebra $\mathfrak{g}$ of the contact symmetry group G. The commutator table is given in Table 5.2 for $1 \le \alpha, \beta, \eta \le 3$ and $2 \le \gamma \le 3$. In this table, when the commutator of two vector fields is generally in the same form of some vector fields in (104), then we have used those general forms again to show the results of commutators.

Theorem 3. *The contact symmetry group of equation (86), is an infinite–dimensional Lie algebra generated by the contact infinitesimal operators (104) with the commutator table 5.2.*

One may repeat the above process for the problem of finding contact Lie algebra of Eq. (86) with the supplementary condition of $\mathbf{n}$ to be unit. In this case there is another condition $v[\text{Eq. (86)}] = 2(n_1\,N_1 + n_2\,N_2 + n_3\,N_3) = 0$ by the action of a contact infinitesimal generator on Eq. (86) which must be added to other relations. Since $\mathbf{n} \ne 0$, so we can suppose $n_1 \ne 0$ and $N_1 = -\frac{n_2}{n_1}\,N_2 - \frac{n_3}{n_1}\,N_3$. Finally we can say

Theorem 4. *The contact symmetry group of equation (86) consisting of unit normal wave front, is an infinite–dimensional Lie algebra and its Lie algebra is generated by the contact infinitesimal operators (103) when we replace the coefficient N_1 by $-\frac{n_2}{n_1}\,N_2 - \frac{n_3}{n_1}\,N_3$. The commutator table of these vector fields is in the form of Table 5.2 when we eliminate the row and column corresponding to v_{N_1} and then change v_{N_2} and v_{N_3} to the new forms.*

6 Summary and conclusions

Towards a deeper understanding of the mysterious behavior of hydrodynamical equations it is necessary to look for various exact solutions. Among these solutions simple waves and multi–waves are the best for compressible flows up to present. These solutions show explicitly as a special case that how it is possible that smooth initial conditions convert to some discontinuities and singularities in future times. Thus, there is a significant hope that by a detailed and deep analysis of these waves one may find more general statements about the appearance of any non-smoothness regarding smooth initial conditions.

In the present work a multidimensional version of simple waves introduced in References [15] and [17] were employed for fully relativistic fluids and plasmas. Each wave front is a plane traveling with its own phase velocity vector. The intersection of different wave fronts is forbidden in the domain of the solution. Also at each instant of time there is a surface as the boundary between the two regions, the region of the validity of the solution and the forbidden region where the solution does not exist. This boundary generally moves and changes in the course of time.

Similar to the nonrelativistic case[17] three essential modes were found, namely vortex, entropy and sound modes. Each mode suggests a wide variety of solutions while only very simple typical solutions were presented as some illustrations of the method of solving. Vortex and entropy modes were solved both in the laboratory and the wave frame. But, due to the high complexity of sound modes we studied them only in the wave frame. Further, as a special physically valid example we considered the thermodynamically state equation at ultra-relativistic temperatures and obtained a complete formal solution in the wave frame.

A symmetry analysis for the vortex mode equation (as a typical equation) led to the finding the structure of point and contact infinitesimal generators as well as fundamental invariants of the equation. In addition a form of general solutions implied by these invariants was obtained. Also we presented some examples for the point transformation case which tend to a precise determination of related symmetry groups. In the special case of our problem, the contact and point symmetry group of the vortex mode equation were both found to be infinite–dimensional Lie groups when the normal vector to wave front is not necessarily unit. When it is of unit length we find a one–dimensional point symmetry group while the contact symmetry group is still infinite–dimensional. The same procedure can be probably made for equations of other modes, namely the entropy mode and the sound mode.

References

[1] J. Smoller, *Shock Waves and Reaction Diffusion Equations* (Springer-Verlag, Berlin, 1983).

[2] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics* second edition (Springer-Verlag, Berlin Heidelberg, 2005).

[3] F. John, *Nonlinear Wave Equations, Formation of Singularities*, Pitcher Lectures in the Mathematical Scinces Held at Lehigh University April 1989 (American Mathematical Society, Providence, 1990).

[4] R. Von Mises, *Mathematical Theory of Compressible Flow* (Academic Press, New York, 1958).

[5] R. Courant and K.O. Friedrichs, *Supersonic Flow and Shock Waves* (Springer-Verlag, New York, 1976).

[6] A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer-Verlag, New York, 1979).

[7] A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko and K.N. Stepanov, *Plasma Electrodynamics* Vol. 1 (Pergamon, Oxford, 1975).

[8] H. Cabannes, *Theoretical Magnetohydrodynamics* (Academic Press, New York, 1970).

[9] B.L. Rozdestvenskii N.N. Janenko, *Systems of Quasilinear Equations and their applications to Gas Dynamics*, Translations of Mathematical Monographs Vol. 55 (American Mathematical Society, Providence, RI, 1980).

[10] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, 1987).

[11] L. Stenflo, A.B. Shvartsburg and J.Weiland, On shock wave formation in a magnetized plasma, *Phys. Lett. A* **225**, 113-116 (1997).

[12] P.K. Shukla, B. Eliasson, M. Marklund and R. Bingham, Nonlinear model for magnetosonic shocklets in plasmas, *Phys. Plasmas* **11**, 2311-2313 (2004).

[13] M. Burnat, The method of Riemann invariants for multi-dimensional nonelliptic systems, *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* **17**, 1019-1026 (1969).

[14] M. Burnat, The method of characteristics and Riemann invariants for multidimensional hyperbolic systems, *Siberian Math. J.* **11**, 210-232 (1970).

[15] G. Boillat, Simple waves in N-dimensional propagation, *J. Math. Phys.* **11**, 1482-1483 (1970).

[16] G.M. Webb, R. Ratkiewicz, M. Brio and G.P. Zank, *Solar Wind 8*, ed. D. Winterhalter, J.T. Gosling, S.R. Habbal, W.S. Kurth and M. Neugebauer, pp 335-338, AIP Conference Proceedings Vol. 382 (American Institute of Physics, New York, 1995).

[17] G.M. Webb, R. Ratkiewicz, M. Brio and G.P. Zank, Multidimensional simple waves in gas dynamics, *J. Plasma Phys.* **59**, 417-460 (1998).

[18] M. Burnat, Geometrical methods in fluid mechanics, *Fluid Dyn. Trans.* **6**, 115-186 (1967).

[19] M. Burnat, The method of solution of hyperbolic systems by means of combining simple waves, *Fluid Dyn. Trans.* **3**, 23-40 (1967).

[20] W. Zajaczkowski, Riemann invariants interaction in MHD double waves, *Demonstratio Math.* **12**, 543-563 (1979).

[21] M. Burnat, Hyperbolic double waves *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* **16**(10) (1968).

[22] Z. Peradzynski, Nonlinear plane k-waves and Riemann invariants, *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* **19**, 625-632 (1971).

[23] Z. Peradzynski, Riemann invariants for the nonplanar k-waves, *Bull. Acad. Polon. Sci., Ser. Sci. Techn.* **19**, 717-724 (1971).

[24] L.V. Komarovskii, An accurate solution of the three-dimensional equations for a nonsteady gas-flow of the double wave type, *Sov. Phys. Dokl.* **135**, 1163–1165 (1960).

[25] D.D. Tskhakaya and H. Eshraghi, Two-dimensional double simple waves in a pair plasma at relativistic temperatures, *Phys. Plasmas* **9**, 2518-2525 (2002).

[26] D.D. Tskhakaya and H. Eshraghi, On the theory of magneto-sound double simple waves, *J. Plasma Phys.* **74**, 455-471 (2008).

[27] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (W.A. Benjamin Inc., New York, Amsterdam, 1967).

[28] A. Lichnerowicz, *Magnetohydrodynamics: Waves and Shock Waves in Curved Space-Time* (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994).

[29] A.M. Anile, *Relativistic Fluids and Magneto-Fluids* (Cambridge University Press, Cambridge, 1989).

[30] I.S. Shikin, Relativistic effects for magnetohydrodynamic waves, *Ann. Inst. Henri Poincare* **11**, 343-372 (1969).

[31] H. Eshraghi, On the vortex dynamics in fully relativistic plsmas, *Phys. Plasmas* **10**, 3577-3583 (2003).

[32] P.J. Olver, *Equivalence, Invariants, and Symmetry* (Cambridge University Press, Cambridge, 1995).

[33] N.H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differentail Equations* (John Wiley & Sons, England, 1999).